Indicative conditionals: algebraic considerations

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Hence, one ought to distinguish between indicative conditionals and counterfactuals.

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It verifies *Adams thesis*:

$$\mathbf{p}(\varphi \to \psi \text{ is true} \mid \varphi \to \psi \text{ has classical value}) = \mathbf{p}(\psi \mid \varphi).$$

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Consider the algebra $\mathbf{DF}_3$ of universe $\{\mathbf{0}, {}^1\!/_2, \mathbf{1}\}$ and operations:

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$\mathbf{0}$	$\mathbf{1}$
$1/2$	$1/2$
$\mathbf{1}$	$\mathbf{0}$

$\wedge_K$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$1/2$	$\mathbf{0}$	$1/2$	$1/2$
$\mathbf{1}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$

$\vee_K$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$1/2$	$1/2$	$1/2$	$\mathbf{1}$
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$\rightarrow_{DF}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
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– Interestingly, we can interchange the language with $\{\neg, \wedge_K, \vee_K, \mathbf{1/2}\}$ by letting

$$x \to_{DF} y := (\mathbf{1/2} \wedge_K \neg x) \vee_K (x \wedge_K y)$$

and, conversely,

$$\mathbf{1/2} := \neg x \to_{DF} (x \to_{DF} x).$$

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– Note that the tables of the $\rightarrow_{\mathrm{DF}}$-free fragment correspond to those of Kleene's logics.

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- $\mathrm{DF}^{\mathbf{1}} := \mathrm{Log}(\mathbf{DF}_3, \{\mathbf{1}\})$ is an expansion of strong Kleene's logic $\mathrm{Log}(\mathbf{K}_3, \{\mathbf{1}\})$.

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- $\mathrm{DF}^{\mathbf{1}} := \mathrm{Log}(\mathbf{DF}_3, \{\mathbf{1}\})$ is an expansion of strong Kleene's logic $\mathrm{Log}(\mathbf{K}_3, \{\mathbf{1}\})$.
- $\mathrm{DF}^{\wedge_K} := \mathrm{Log}[(\mathbf{DF}_3, \{{}^1\!/\!_2, \mathbf{1}\}), (\mathbf{DF}_3, \{\mathbf{1}\})]$ is an expansion of Kleene's logic of order $\mathrm{Log}[(\mathbf{K}_3, \{{}^1\!/\!_2, \mathbf{1}\}), (\mathbf{K}_3, \{\mathbf{1}\})]$.

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- $\mathrm{DF}^{\wedge_K}$ can be axiomatized by adding the axioms

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- DF can be axiomatized by replacing the disjunctive syllogism rule with the excluded middle:

  $$\vdash \varphi \vee \neg\varphi.$$

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– For every De Finettian logic $\vdash$ it holds that

$$\mathrm{Alg}^*(\vdash) \subsetneq \mathrm{Alg}(\vdash) = \mathbf{CK},$$

where **CK** is the variety of centered Kleene lattices.

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(i) the former sanctions intuitively false implications:

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(ii) the latter sanctions classically false implications:

2.19 *If the water supply is off then she cannot cook dinner, or if the gas supply is off then she cannot cook dinner. Therefore, if either the water supply or the gas supply is off, then she cannot cook dinner.*

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Let $\mathrm{sOL} := \mathrm{Log}(\mathbf{O}_3, \{\mathbf{1/2}, \mathbf{1}\})$. Then, $\mathrm{OL} = \mathrm{sOL} + p \wedge \neg p \vdash q$.

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- sOL can be axiomatized by extending the classical $\{\wedge, \vee\}$-fragment with axioms
  - i. $\neg\neg\varphi \leftrightarrow \varphi$,
  - ii. $(\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$,
  - iii. $\neg(\varphi \wedge \psi) \leftrightarrow ((\varphi \rightarrow \neg\psi) \wedge (\psi \rightarrow \neg\varphi))$,
  - iv. $\neg(\varphi \rightarrow \psi) \leftrightarrow (\varphi \rightarrow \neg\psi)$ (*Boethius theses*).

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- The equivalent semantics of sOL is the variety $\mathbf{V}(\mathbf{OL}_3)$. Its elements are called *OL-algebras*.
- *OL-twist algebras* can be used in order to prove some results.
- If we add one of the constants $\mathbf{0}, \mathbf{1}$ to the language of $\mathbf{OL}_3$, it turns to be primal (hence, we obtain functional completeness).

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- Farrell-Cantwell logic is also a subsystem of sOL (see next slides).

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iii. The one having $\wedge_{\mathsf{K}}$ as meet: $\mathbf{0} < {}^{\mathbf{1}}\!/_{\mathbf{2}} < \mathbf{1}$.

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Also: what about their axiomatizations? Perhaps we need Set-Set calculi and non-deterministic matrices?

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$\mathbf{F}_3$: $\neg, \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}$ with

$\to_{\mathsf{F}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$
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$^1/_2$	$\mathbf{0}$	$^1/_2$	$^1/_2$
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$\mathbf{CN}_3$: $\neg, \wedge_K, \vee_K$ with

$\to_{OL}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$1/2$	$1/2$	$1/2$
$1/2$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$

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Then:

– Both logics are definitionally equivalent letting

$$x \to_{\mathsf{OL}} y := \neg((y \to_{\mathsf{F}} x) \to_{\mathsf{F}} \neg y) \to_{\mathsf{F}} ((x \vee_{\mathsf{K}} y) \to_{\mathsf{F}} y),$$

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- Both logics are algebraizable and axiomatizable (CN is an extension of Wansing's connexive logic C).

- Both logics are subsystems of sOL.

Future work

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4. Get back to the philosophical motivations of these logics.

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